

# On the Titchmarsh convolution theorem for distributions on a circle

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## Abstract

We prove a version of the Titchmarsh convolution theorem for distributions supported in several intervals of the circle. We show that the “naïve form” of the Titchmarsh theorem could be violated, but that such a violation is only possible for the convolution of distributions which both possess certain symmetry properties.

KEYWORDS: Titchmarsh convolution theorem, symmetry properties.

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## 1 Introduction

The Titchmarsh convolution theorem [Tit26] states that for any two compactly supported distributions  $f, g \in \mathcal{E}'(\mathbb{R})$ ,

$$\inf \text{supp } f * g = \inf \text{supp } f + \inf \text{supp } g, \quad \sup \text{supp } f * g = \sup \text{supp } f + \sup \text{supp } g. \quad (1.1)$$

The higher-dimensional reformulation by Lions [Lio51] states that for  $f, g \in \mathcal{E}'(\mathbb{R}^n)$ , the convex hull of the support of  $f * g$  is equal to the sum of convex hulls of supports of  $f$  and  $g$ . Different proofs of the Titchmarsh convolution theorem are contained in [Yos80, Chapter VI] (Real Analysis style), [Hör90, Theorem 4.3.3] (Harmonic Analysis style), and [Lev96, Lecture 16, Theorem 5] (Complex Analysis style).

In this note, we generalize the Titchmarsh Theorem for distributions on the circle  $\mathbb{T} = \mathbb{R} \setminus 2\pi$ .

First, we note that there are zero divisors with respect to the convolution on a circle. Indeed, for any two distributions  $f, g \in \mathcal{E}'(\mathbb{T})$  one has

$$(f + S_\pi f) * (g - S_\pi g) = f * g + S_\pi(f * g) - S_\pi(f * g) - f * g = 0. \quad (1.2)$$

Above,  $S_y, y \in \mathbb{T}$ , is the shift operator, defined on  $\mathcal{E}'(\mathbb{T})$  by

$$(S_y f)(\omega) = f(\omega - y), \quad (1.3)$$

where the above relation is understood in the sense of distributions. Yet, the cases when the Titchmarsh convolution theorem “does not hold” (in some naïve form) could be specified. This leads to a version of the Titchmarsh convolution theorem for distributions on a circle (Theorem 1 below).

Our interest in properties of a convolution on a circle is due to applications to the theory of attractors for finite difference approximations of nonlinear dispersive equations. In [KK07], we considered the weak attractor of finite energy solutions to the  $U(1)$ -invariant Klein-Gordon equation in 1D, coupled to a nonlinear oscillator. We proved that the global attractor of all finite energy solutions is formed by the set of all solitary waves,  $\phi_\omega(x)e^{-i\omega t}$  with  $\omega \in \mathbb{R}$  and  $\phi_\omega \in H^1(\mathbb{R})$ . The general strategy of the proof was to consider the omega-limit trajectories of the finite energy solution  $\psi(x, t) \in \mathbb{C}$ , defined as solutions with the Cauchy data at the omega-limit points of the set  $\{(\psi(t), \dot{\psi}(t)): t \geq 0\}$  in the local energy

seminorms. One shows that the time spectrum of each omega-limit trajectory is inside the spectral gap and then, applying the Titchmarsh convolution theorem to the equation satisfied by the omega-limit trajectory, one concludes that its time spectrum consists of at most a single frequency, hence any omega-limit trajectory is a solitary wave (or zero). We expect to parallel this approach for the Klein-Gordon equation in discrete space-time [SV78]. The main difference is that now the frequency domain is a circle (no longer the whole real line) and there are not one, but two spectral gaps in the continuous spectrum. Thus, the convolution theorem is to be applied to distributions with the frequency support in the two spectral gaps. To analyze the time spectrum of the omega-limit trajectory, one needs a version of the Titchmarsh convolution theorem for distributions supported on two intervals of the circle.

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## 2 Main results

For  $I \subset \mathbb{T}$  and  $n \in \mathbb{N}$ , denote

$$\mathcal{R}_n(I) = \bigcup_{k \in \mathbb{Z}_n} S_{\frac{2\pi k}{n}} I, \quad \text{where } \mathbb{Z}_n = \mathbb{Z} \bmod n.$$

Let  $f, g \in \mathcal{E}'(\mathbb{T})$ . Let  $I, J \subset \mathbb{T}$  be two closed intervals such that  $\text{supp } f \subset \mathcal{R}_n(I)$ ,  $\text{supp } g \subset \mathcal{R}_n(J)$ , and assume that there is no closed interval  $I' \subsetneq I$  such that  $\text{supp } f \subset \mathcal{R}_n(I')$  and no closed interval  $J' \subsetneq J$  such that  $\text{supp } g \subset \mathcal{R}_n(J')$ .

**Remark 1.** For  $f, g \in \mathcal{E}'(\mathbb{T})$ , the intervals  $I$  and  $J$  play the role similar to “convex hulls” of supports.

**Theorem 1** (Titchmarsh theorem for distributions on a circle). *Let  $n \in \mathbb{N}$ ,  $n \geq 2$ . Assume that*

$$|I| + |J| < \frac{2\pi}{n}. \quad (2.1)$$

*Let  $K \subset I + J \subset \mathbb{T}$  be a closed interval such that  $\text{supp } f * g \subset \mathcal{R}_n(K)$ . If  $\lambda := \inf K - \inf I - \inf J > 0$ , then there are  $\alpha, \beta \in \mathbb{C}$ ,  $\alpha^n = \beta^n = 1$ ,  $\alpha \neq \beta$ , such that*

$$\left( \sum_{k \in \mathbb{Z}_n} \alpha^k S_{\frac{2\pi k}{n}} f \right) \Big|_{(\sup I - \frac{2\pi}{n}, \inf I + \lambda)} = 0, \quad \inf \text{supp} \left( \sum_{k \in \mathbb{Z}_n} \alpha^k S_{\frac{2\pi k}{n}} g \right) \Big|_{(\sup J - \frac{2\pi}{n}, \inf J + \lambda)} = \inf J, \quad (2.2)$$

$$\inf \text{supp} \left( \sum_{k \in \mathbb{Z}_n} \beta^k S_{\frac{2\pi k}{n}} f \right) \Big|_{(\sup I - \frac{2\pi}{n}, \inf I + \lambda)} = \inf I, \quad \left( \sum_{k \in \mathbb{Z}_n} \beta^k S_{\frac{2\pi k}{n}} g \right) \Big|_{(\sup J - \frac{2\pi}{n}, \inf J + \lambda)} = 0. \quad (2.3)$$

**Remark 2.** The relations (2.3) follow from (2.2) due to the symmetric role of  $f$  and  $g$ . The conclusion  $\alpha \neq \beta$  follows from comparing (2.2) and (2.3). Indeed, the first relation in (2.2) implies that  $\inf \text{supp} \left( \sum_{k \in \mathbb{Z}_n} \alpha^k S_{\frac{2\pi k}{n}} f \right) \Big|_I \geq \inf I + \lambda > \inf I$ , which would contradict the first relation in (2.3) if we had  $\alpha = \beta$ .

Applying the reflection to  $\mathbb{T}$ , we also get the following result:

**Corollary 1.** *If  $\rho := \sup I + \sup J - \sup K > 0$ , then there are  $\alpha, \beta \in \mathbb{C}$ ,  $\alpha^n = \beta^n = 1$ ,  $\alpha \neq \beta$ , such that*

$$\left( \sum_{k \in \mathbb{Z}_n} \alpha^k S_{\frac{2\pi k}{n}} f \right) \Big|_{(\sup I - \rho, \inf I + \frac{2\pi}{n})} = 0, \quad \sup \text{supp} \left( \sum_{k \in \mathbb{Z}_n} \alpha^k S_{\frac{2\pi k}{n}} g \right) \Big|_{(\sup J - \rho, \inf J + \frac{2\pi}{n})} = \sup J, \quad (2.4)$$

$$\sup \text{supp} \left( \sum_{k \in \mathbb{Z}_n} \beta^k S_{\frac{2\pi k}{n}} f \right) \Big|_{(\sup I - \rho, \inf I + \frac{2\pi}{n})} = \sup I, \quad \left( \sum_{k \in \mathbb{Z}_n} \beta^k S_{\frac{2\pi k}{n}} g \right) \Big|_{(\sup J - \rho, \inf J + \frac{2\pi}{n})} = 0. \quad (2.5)$$

That is, if  $K \subsetneq I + J$  (informally, we could say that certain naïve form of the Titchmarsh convolution theorem is not satisfied), then both  $f$  and  $g$  satisfy certain symmetry properties on  $\mathcal{R}_n(U)$  and on  $\mathcal{R}_n(V)$ , where open non-intersecting intervals  $U$  and  $V$  can be chosen so that  $U \cup K \cup V \supset I + J$ .

In the case  $n = 2$ , we have the following result.

**Corollary 2.** *Let  $n = 2$ ,  $f, g \in \mathcal{E}'(\mathbb{T})$ , and let  $I, J, K$  be as in Theorem 1. Then  $\lambda := \inf K - \inf I - \inf J > 0$  if and only if there is  $\alpha = \pm 1$  such that*

$$(f + \alpha S_\pi f) \Big|_{(\sup I - \pi, \inf I + \lambda)} = 0, \quad (g - \alpha S_\pi g) \Big|_{(\sup J - \pi, \inf J + \lambda)} = 0.$$

*Proof of Corollary 2.* The “only if” part follows from Theorem 1. We check the “if” part by direct computation. Let  $f \in \mathcal{E}'(I \cup S_\pi I)$ , where  $I \subset \mathbb{T}$ ,  $|I| < \pi/2$ ,  $g \in \mathcal{E}'(J \cup S_\pi J)$ , where  $J \subset \mathbb{T}$ ,  $|J| < \pi/2$ , and assume that  $f = \pm S_\pi f$  on  $(\sup I - \pi, \inf I + \lambda)$ ,  $g = \mp S_\pi g$  on  $(\sup J - \pi, \inf J + \lambda)$ . Then, as in (1.2),

$$\begin{aligned} & (f * g)|_{(\sup I + \sup J - 2\pi, \inf I + \inf J + \lambda)} \\ &= f|_{(\sup I - \pi, \inf I + \lambda)} * g|_{(\sup J - \pi, \inf J + \lambda)} + (S_\pi f)|_{(\sup I - \pi, \inf I + \lambda)} * (S_\pi g)|_{(\sup J - \pi, \inf J + \lambda)} \\ &= f|_{(\sup I - \pi, \inf I + \lambda)} * g|_{(\sup J - \pi, \inf J + \lambda)} - f|_{(\sup I - \pi, \inf I + \lambda)} * g|_{(\sup J - \pi, \inf J + \lambda)} = 0. \end{aligned}$$

□

Denote  $f^\#(\omega) = \overline{f(-\omega)}$ . Let  $f \in \mathcal{E}'(\mathbb{T})$  and let  $I \subset \mathbb{T}$  be a closed interval such that  $\text{supp } f \subset \mathcal{R}_2(I)$ . Assume that there is no closed interval  $I' \subsetneq I$  such that  $\text{supp } f \subset \mathcal{R}_2(I')$ .

**Theorem 2.** *If  $I \subset (-\pi/2, \pi/2)$  and  $|I| < \pi/2$ , then the inclusion  $\text{supp } f * f^\# \subset \{0; \pi\}$  implies that  $\text{supp } f \subset \{\inf I; \sup I; \pi + \inf I; \pi + \sup I\}$ . Moreover, there are distributions  $\mu, \nu \in \mathcal{E}'(\mathbb{T})$ , each supported at a point, such that*

$$f = \mu + S_\pi \mu + \nu - S_\pi \nu. \quad (2.6)$$

Finally, let us also formulate the convolution theorem for powers of a distribution. Let  $f \in \mathcal{E}'(\mathbb{T})$ . Let  $I \subset \mathbb{T}$  be a closed interval such that  $\text{supp } f \subset \mathcal{R}_n(I)$ , and assume that there is no  $I' \subsetneq I$  such that  $\text{supp } f \subset \mathcal{R}_n(I')$ .

**Theorem 3** (Titchmarsh theorem for powers of a distribution on a circle). *Assume that  $|I| < \frac{2\pi}{pn}$ , for some  $p \in \mathbb{N}$ . Then the smallest closed interval  $K \subset pI$  such that  $\text{supp } f^{*p} \subset \mathcal{R}_n(K)$  is  $K = pI$ .*

Above, we used the notations  $pI = \underbrace{I + \dots + I}_p$  and  $f^{*p} = \underbrace{f * \dots * f}_p$ .

### 3 Proofs

First, we prove the following lemma.

**Lemma 1.** *Let  $f_j \in \mathcal{E}'(I)$ ,  $j \in \mathbb{Z}_n$ . There is  $\alpha \in \mathbb{C}$ ,  $\alpha^n = 1$ , such that*

$$\inf \text{supp } \sum_{j \in \mathbb{Z}_n} \alpha^j f_j = \min_{j \in \mathbb{Z}_n} \inf \text{supp } f_j. \quad (3.1)$$

*Proof.* Denote  $a := \min_{j \in \mathbb{Z}_n} \inf \text{supp } f_j$ . Let us assume that, contrary to the statement of the lemma, there is  $\epsilon > 0$  such that  $\inf \text{supp } \sum_{j \in \mathbb{Z}_n} \alpha^j f_j \geq a + \epsilon$ , for any  $\alpha = \gamma^m$ , where  $\gamma = \exp(\frac{2\pi i}{n})$  and  $m \in \mathbb{N}$ ,  $1 \leq m \leq n$ . Then for any test function  $\varphi \in \mathcal{D}(\mathbb{R})$  with  $\text{supp } \varphi \subset (a - \epsilon, a + \epsilon)$  we would have:

$$0 = \langle \varphi, \sum_{j \in \mathbb{Z}_n} \gamma^{jm} f_j \rangle = \sum_{j \in \mathbb{Z}_n} \gamma^{jm} \langle \varphi, f_j \rangle, \quad 1 \leq m \leq n. \quad (3.2)$$

Using the formula for the Vandermonde determinant, we have

$$\det \begin{bmatrix} 1 & \gamma & \gamma^2 & \dots & \gamma^{n-1} \\ 1 & \gamma^2 & \gamma^4 & \dots & \gamma^{2(n-1)} \\ 1 & \gamma^3 & \gamma^6 & \dots & \gamma^{3(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \gamma^n & \gamma^{2n} & \dots & \gamma^{n(n-1)} \end{bmatrix} = \prod_{1 \leq j < k \leq n} (\gamma^k - \gamma^j) \neq 0. \quad (3.3)$$

Hence, (3.2) implies that  $\langle \varphi, f_j \rangle = 0$  for all  $j \in \mathbb{Z}_n$ . Due to arbitrariness of  $\varphi$ , this leads to  $f_j|_{(a-\epsilon, a+\epsilon)} = 0$  for all  $j \in \mathbb{Z}_n$ , leading to a contradiction with the definition of  $a$ . □

*Proof of Theorem 1.* One has  $\text{supp } f \subset \mathcal{R}_n(I)$ ,  $\text{supp } g \subset \mathcal{R}_n(J)$ ,  $\text{supp } f * g \subset \mathcal{R}_n(K) \subset \mathcal{R}_n(I + J)$ . Due to the restriction (2.1), the sets  $\mathcal{R}_n(I)$ ,  $\mathcal{R}_n(J)$ , and  $\mathcal{R}_n(I + J)$  each consist of  $n$  non-intersecting intervals. For  $j \in \mathbb{Z}_n$ , let us set  $f_j = (S_{\frac{2\pi j}{n}} f)|_I \in \mathcal{E}'(I)$ ,  $g_j = (S_{\frac{2\pi j}{n}} g)|_J \in \mathcal{E}'(J)$ ,  $h_j = (S_{\frac{2\pi j}{n}}(f * g))|_K \in \mathcal{E}'(I + J)$ ; then

$$h_j = (S_{\frac{2\pi j}{n}}(f * g))|_{I+J} = \sum_{\substack{k+l=j \\ k, l \in \mathbb{Z}_n}} (S_{\frac{2\pi k}{n}} f)|_I * (S_{\frac{2\pi l}{n}} g)|_J = \sum_{\substack{k+l=j \\ k, l \in \mathbb{Z}_n}} f_k * g_l, \quad j \in \mathbb{Z}_n. \quad (3.4)$$

Using the relation (3.4), for any  $\alpha \in \mathbb{C}$  such that  $\alpha^n = 1$  we have:

$$\left( \sum_{k \in \mathbb{Z}_n} \alpha^k f_k \right) * \left( \sum_{l \in \mathbb{Z}_n} \alpha^l g_l \right) = \sum_{j \in \mathbb{Z}_n} \alpha^j \left[ \sum_{\substack{k+l=j \\ k, l \in \mathbb{Z}_n}} f_k * g_l \right] = \sum_{j \in \mathbb{Z}_n} \alpha^j h_j. \quad (3.5)$$

Applying the Titchmarsh convolution theorem (1.1) to this relation, we obtain:

$$\inf \text{supp} \sum_{j \in \mathbb{Z}_n} \alpha^j f_j + \inf \text{supp} \sum_{j \in \mathbb{Z}_n} \alpha^j g_j = \inf \text{supp} \sum_{j \in \mathbb{Z}_n} \alpha^j h_j \geq \inf K, \quad (3.6)$$

where we took into account that  $\min_{j \in \mathbb{Z}_n} \inf \text{supp} h_j \geq \inf K$ . By Lemma 1, there is  $\alpha \in \mathbb{C}$ ,  $\alpha^n = 1$ , such that  $\inf \text{supp} \sum_{j \in \mathbb{Z}_n} \alpha^j g_j = \min_{j \in \mathbb{Z}_n} \inf \text{supp} g_j = \inf J$ ; this is equivalent to the second relation in (2.2). For this value of  $\alpha$ , (3.6) yields:

$$\inf \text{supp} \sum_{j \in \mathbb{Z}_n} \alpha^j f_j \geq \inf K - \inf J = \inf I + \lambda.$$

This is equivalent to the first relation in (2.2). According to Remark 2, this finishes the proof.  $\square$

*Proof of Theorem 2.* If  $I$  consists of one point,  $I = \{p\} \subset (-\pi/2, \pi/2)$ , then  $\text{supp } f = \mathcal{R}(p) = \{p; \pi + p\}$ , and (2.6) holds with

$$\mu = \frac{f + S_\pi f}{2} \Big|_I, \quad \nu = \frac{f - S_\pi f}{2} \Big|_I.$$

Now we assume that  $|I| > 0$ . Define  $J = -I$  and  $K = \{0\} \subset I + J$ . Then  $\text{supp } f^\# \subset \mathcal{R}_2(J)$  and there is no  $J' \subsetneq J$  such that  $\text{supp } f^\# \subset \mathcal{R}_2(J')$ . According to the conditions of the theorem,  $\text{supp } f * f^\# \subset \mathcal{R}_2(K)$ ; hence, one has:

$$\lambda := \inf K - \inf I - \inf J = \sup I - \inf I = |I| > 0. \quad (3.7)$$

Applying Theorem 1 to (3.7), we conclude that there is  $\alpha \in \{\pm 1\}$  such that

$$(f + \alpha S_\pi f)|_{(\sup I - \pi, \sup I)} f = 0 \quad (3.8)$$

and also  $\inf \text{supp}(f^\# + \alpha S_\pi f^\#)|_{(-\frac{\pi}{2}, \frac{\pi}{2})} = -\sup I$ ; this last relation implies that

$$\sup \text{supp}(f + \alpha S_\pi f)|_{(-\frac{\pi}{2}, \frac{\pi}{2})} = \sup I. \quad (3.9)$$

Similarly, by Theorem 1, there is  $\beta \in \{\pm 1\}$  such that  $(f^\# + \beta S_\pi f^\#)|_{(-\inf I - \pi, -\inf I)} = 0$ , hence

$$(f + \beta S_\pi f)|_{(\inf I, \inf I + \pi)} = 0. \quad (3.10)$$

Comparing (3.9) with (3.10), we conclude that  $\alpha \neq \beta$ , hence  $\alpha = -\beta$ ; then (3.8) and (3.10) allow us to conclude that both  $f$  and  $S_\pi f$  vanish on  $(\inf I, \sup I)$ , hence

$$\text{supp } f \subset \{\inf I; \sup I; \pi + \inf I; \pi + \sup I\}.$$

By (3.8) and (3.10), if  $\alpha = 1$ , the relation (2.6) holds with  $\mu = f|_{(\inf I, \pi/2)}$  and  $\nu = f|_{(-\pi/2, \sup I)}$ . If instead  $\alpha = -1$ , the relation (2.6) holds with  $\mu = f|_{(-\pi/2, \sup I)}$  and  $\nu = f|_{(\inf I, \pi/2)}$ .  $\square$

Let us notice that the proof of Theorem 3 for the case  $p = 2$  immediately follows from Theorem 1. (For example, the relations (2.2) with  $f = g$  are mutually contradictory unless  $\lambda = 0$ .) By induction, this also gives the proof for  $p = 2^N$ , with any  $N \in \mathbb{N}$ . Then one can deduce the statement of Theorem 3 for any  $p \leq 2^N$ , but under the condition  $|I| < \frac{2\pi}{2^N n}$ , which is stronger than  $|I| < \frac{2\pi}{pn}$ . Instead of trying to use Theorem 1, we give an independent proof.

*Proof of Theorem 3.* One has  $\text{supp } f^{*p} \subset \mathcal{R}_n(pI)$ . Due to the smallness of  $I$ , both  $\mathcal{R}_n(I)$  and  $\mathcal{R}_n(pI)$  are collections of  $n$  non-intersecting intervals. Define  $f_j := (S_{\frac{2\pi j}{n}} f)|_I \in \mathcal{E}'(I)$  and  $h_j := (S_{\frac{2\pi j}{n}}(f^{*p}))|_I \in \mathcal{E}'(I)$ . Then

$$h_j = (S_{\frac{2\pi j}{n}}(f^{*p}))|_{pI} = \sum_{\substack{j_1 + \dots + j_p = j \\ j_1, \dots, j_p \in \mathbb{Z}_n}} (S_{\frac{2\pi j_1}{n}} f)|_I * \dots * (S_{\frac{2\pi j_p}{n}} f)|_I = \sum_{\substack{j_1 + \dots + j_p = j \\ j_1, \dots, j_p \in \mathbb{Z}_n}} f_{j_1} * \dots * f_{j_p}, \quad j \in \mathbb{Z}_n. \quad (3.11)$$

Taking into account (3.11), for any  $\alpha \in \mathbb{C}$  such that  $\alpha^n = 1$  one has:

$$\left( \sum_{j \in \mathbb{Z}_n} \alpha^j f_j \right)^{*p} = \sum_{j \in \mathbb{Z}_n} \alpha^j \left[ \sum_{j_1 + \dots + j_p = j \pmod n} f_{j_1} * \dots * f_{j_p} \right] = \sum_{j \in \mathbb{Z}_n} \alpha^j h_j. \quad (3.12)$$

Now we apply the Titchmarsh convolution theorem to (3.12), getting

$$p \inf \text{supp} \sum_{j \in \mathbb{Z}_n} \alpha^j f_j = \inf \text{supp} \sum_{j \in \mathbb{Z}_n} \alpha^j h_j.$$

By Lemma 1, there is  $\alpha \in \mathbb{C}$ ,  $\alpha^n = 1$ , such that  $\inf \text{supp} \sum_{j \in \mathbb{Z}_n} \alpha^j f_j = \min_{j \in \mathbb{Z}_n} \inf \text{supp} f_j$ , hence, for this value of  $\alpha$ ,

$$p \min_{j \in \mathbb{Z}_n} \inf \text{supp} f_j = \inf \text{supp} \sum_{j \in \mathbb{Z}_n} \alpha^j h_j \geq \min_{j \in \mathbb{Z}_n} \inf \text{supp} h_j.$$

On the other hand, (3.11) immediately yields the inequalities  $\inf \text{supp} h_j \geq p \min_{k \in \mathbb{Z}_n} \inf \text{supp} f_k$ , for any  $j \in \mathbb{Z}_n$ . It follows that  $\min_{j \in \mathbb{Z}_n} \inf \text{supp} h_j = p \min_{j \in \mathbb{Z}_n} \inf \text{supp} f_j$  and similarly  $\max_{j \in \mathbb{Z}_n} \sup \text{supp} h_j = p \max_{j \in \mathbb{Z}_n} \sup \text{supp} f_j$ .  $\square$

## References

- [Hör90] L. Hörmander, *The analysis of linear partial differential operators. I*, Springer Study Edition, Springer-Verlag, Berlin, 1990, second edn.
- [KK07] A. I. Komech and A. A. Komech, *Global attractor for a nonlinear oscillator coupled to the Klein-Gordon field*, Arch. Ration. Mech. Anal. **185** (2007), pp. 105–142.
- [Lev96] B. Y. Levin, *Lectures on entire functions*, vol. 150 of *Translations of Mathematical Monographs*, American Mathematical Society, Providence, RI, 1996, in collaboration with and with a preface by Yu. Lyubarskii, M. Sodin and V. Tkachenko, Translated from the Russian manuscript by Tkachenko.
- [Lio51] J.-L. Lions, *Supports de produits de composition. I*, C. R. Acad. Sci. Paris **232** (1951), pp. 1530–1532.
- [SV78] W. Strauss and L. Vazquez, *Numerical solution of a nonlinear Klein-Gordon equation*, J. Comput. Phys. **28** (1978), pp. 271–278.
- [Tit26] E. Titchmarsh, *The zeros of certain integral functions*, Proc. of the London Math. Soc. **25** (1926), pp. 283–302.
- [Yos80] K. Yosida, *Functional analysis*, vol. 123 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, Springer-Verlag, Berlin, 1980, sixth edn.